Introduction to Differential Equations Without the Agonizing Pain: Practice Solutions

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I. First-order ordinary differential equations

(1) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$

- Tip: Separate the variables and integrate both sides to solve the differential equation.

(2) $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}$

- Tip: Rewrite this in standard form, find appropriate integration factors, express the left-hand side as a differential, and integrate both sides of the equation.

(3) $\frac{\mathrm{d}y}{\mathrm{d}x} + (x - 2y) = 0$

- Tip: Check if it is an exact equation. If not, find the integrating factor and convert it into the exact equation.

Solutions

(1)

We can separate variables and integrate both sides with respect to their respective variables.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x \\ \int \mathrm{d}y &= \int 2x \,\mathrm{d}x \\ \int \mathrm{d}y &= \frac{1}{2} \cdot 2x^{(1+1)} + C \\ y &= x^2 + C\end{aligned}$$

(2)

First, we can rewrite in Standard Form $\frac{\mathrm{d}x}{\mathrm{d}y} + P(x) = Q(x)$, where $P(x) = 2$, and find the potential function $\phi(x) = e^{\int P(x)\mathrm{d}x} = e^{\int 2\mathrm{d}x}$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= e^{-x}\\ e^{\int 2\mathrm{d}x}(\frac{\mathrm{d}y}{\mathrm{d}x} + 2y) &= e^{\int 2\mathrm{d}x}e^{-x}\\ e^{\int 2\mathrm{d}x}\frac{\mathrm{d}y}{\mathrm{d}x} + e^{\int 2\mathrm{d}x}2y &= 1\end{aligned}$$

Next, we can express the left-hand side as differential:

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{2x}y)\mathrm{dx} = 1$$

Finally, integrate both sides and solve for y:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(e^{2x}y)\mathrm{dx} &= 1\\ \int \frac{\mathrm{d}}{\mathrm{d}x}(e^{2x}y)\mathrm{dx} &= \int 1\ \mathrm{dx}\\ e^{2x}y &= e^{x} + C\\ y &= e^{-x} + Ce^{-2x}\end{aligned}$$

(3)

First we rewrite the equation:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + (x - 2y) &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} - 2y &= -x\end{aligned}$$

The equation is exact if the partial derivatives with respect to y of the coefficient of $\mathrm{d}x$ and with respect to x of the coefficient of $\mathrm{d}x$ are equal.

$$\begin{aligned}M_x &= \frac{\partial}{\partial x}1 = 0\\ N_y &= \frac{\partial}{\partial y} - 2y = -2\end{aligned}$$

$$0 \neq -2$$

Since $M_x \neq N_y$, the equation is not exact. The integrating factor μ can be found using $\mu = e^{\int \frac{N_y - M_x}{M} \mathrm{d}x}$. In this case:

$$\mu = e^{\int \frac{N_y - M_x}{M} \mathrm{d}x} = e^{\int -2 \mathrm{d}x} = e^{-2x}$$

Multiply the entire equation by μ:

$$\begin{aligned} \mu M(x,y)\mathrm{d}x + \mu N(x,y)\mathrm{d}y &= -x \\ e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y &= -e^{-2x}x \\ e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}(e^{-2x})y &= -e^{-2x}x \\ \frac{\mathrm{d}}{\mathrm{d}x}(e^{-2x}y) &= -e^{-2x}x \\ y &= \frac{x}{2} + \frac{1}{4} + Ce^{2x} \end{aligned}$$

II. Linear differential equations with constant coefficients

(1) Find the roots of the characteristic equation for $y'' + 2y' + 2y = 0$ and write down its solution.
(2) Using the Exponential-input Theorem to solve $\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^x$.
(3) Expand the Laplace transform of the function $f(x) = e^{2x}$ over the interval $[0, \infty]$.
(4) Find the inverse function of the Laplace transform for the function $F(s) = \frac{1}{s^2+4s+5}$.

Solutions

(1)

To find the roots of the characteristic equation for the given second-order linear homogeneous differential equation $y'' + 2y' + 2y = 0$, we can write down the characteristic equation by replacing the derivatives with the corresponding terms.

The characteristic equation is obtained by substituting $y = e^r t$ into the differential equation:

$$r^2 + 2r + 2 = 0$$

For the given equation, $a = 1, b = 2, c = 2$ .The solutions can be found using the quadratic formula:

$$\Delta = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 2 = -4$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm \sqrt{-4}}{2}$$

Since the Δ is negative, the roots will be complex conjugates:

$$r = -1 \pm i$$

That's $\alpha = -1, \beta = 1$. According to Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, we can substitute:

$$e^{(\alpha + i\beta)x} = e^{\alpha x}(\cos(\beta x) \pm i\sin(\beta x))$$

So we got the general solution formula:

$$\begin{aligned} y &= C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ &= C_1 e^{\alpha x}(\cos(\beta x) + i\sin(\beta x)) + C_2 e^{\alpha x}(\cos(\beta x) - i\sin(\beta x)) \end{aligned}$$

Finally, by substituting, we got the solution of the equation:

$$y = e^{-x}(C_1 \sin(x) + C_2 \cos(x))$$

(2)

We can rewrite in Standard Form $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x) = Q(x)$, where $P(x), Q(x)$ are function of x, and find the potential function $\phi(x) = e^{\int P(x)\mathrm{d}x} = e^{\int 1\mathrm{d}x} = e^x$

$$\begin{aligned} e^x \frac{\mathrm{d}y}{\mathrm{d}x} + e^x y &= e^{2x} \\ \frac{\mathrm{d}}{\mathrm{d}y}(e^x y) &= e^{2x} \\ \int \frac{\mathrm{d}}{\mathrm{d}y}(e^x y)\mathrm{d}x &= \int e^{2x}\mathrm{d}x \\ e^x y &= \frac{1}{2}e^{2x} + C \\ y &= \frac{1}{2}e^{2x} + Ce^{-x} \end{aligned}$$

(3)

The Laplace transform of a function $f(x)$ defined on $[0, \infty)$ is given by the integral:

$$\mathcal{L}\{f(x)\} = \int_0^\infty f(x) e^{-sx} \mathrm{d}x$$

In the case:

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \int_0^\infty e^{2x} e^{-sx} \mathrm{d}x \\ &= \int_0^\infty e^{(2-s)x} \mathrm{d}x\end{aligned}$$

To find the Laplace transform we need to compute the integral, the result of which will depend on the complex variable s. The Laplace transform is only defined when the real part of s is greater than the real part of the poles of the function (That's $\Re(s) > 2$).

The integral is given by:

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \lim_{a\to\infty} \left[\frac{e^{(2-s)x}}{2-s}\right]_0^a \\ &= \lim_{a\to\infty} \frac{e^{(2-s)a} - 1}{2-s}\end{aligned}$$

if $\Re(s) > 2,\ 2 - \Re(s)$ is negative and as a approaches infinity, the exponential term $e^{(2-s)a}$ goes to 0, the limit becomes

$$\mathcal{L}\{f(x)\} = \lim_{a\to\infty} \frac{-1}{2-s} = \frac{1}{s-2}$$

(4)

We first need to express $F(s)$ in partial fraction form and then find the inverse transforms of each term:

$$F(s) = \frac{1}{s^2 + 4s + 5} = \frac{1}{(s+2)^2 + 1}$$

Now, we can use the Laplace transform pair to find the inverse Laplace transform:

$$\mathcal{L}^{-1}\{e^{-ax}\sin(bx)\} = \frac{b}{(s+a)^2+b^2}$$

That's $a = 2, b = 1$:

$$\mathcal{L}^{-1}\{f(x)\} = e^{-ax}\sin(bx) = e^{-2x}\sin(x)$$

III. Numerical Methods

(1) Write a function that takes a list as input and returns the difference between each element and its succeeding element in the list.
(2) Write a function that takes a tuple representing an interval and solves the ordinary differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y$ with a step size of $h = 0.1$.

- Errata: In addition, the function also needs to accept a tuple representing the initial value

Solutions (Haskell)

(1)

I've received some inquiries from readers regarding this question. They thought it involved providing a function expression for analysis and then receiving a set of discrete x inputs to calculate the differences. In reality, my intention is simply to calculate the differences between adjacent elements in a sequence.

```
forwardDifference :: Num a => [a] -> [a]
forwardDifference [] = []
forwardDifference [_] = []
forwardDifference (x:y:xs) = (y - x) : forwardDifference (y:xs)
```

Example for Test:

```
main :: IO ()
main = do
  let inputList = [1, 4, 7, 11, 16]
      diffList = forwardDifference inputList
  putStrLn $ "Input List: " ++ show inputList
  putStrLn $ "Forward Differences: " ++ show diffList
```

Output: [3,3,4,5]

(2)

We define a function representing the right-hand side of the ordinary differential equation, and within the Euler method function, we take an initial condition (x, y), a solution interval (a, b), and a step size h. We use the iterate function to construct an infinite list, where each element is the result of applying the Euler method to obtain the next point.

```
equation :: Double -> Double -> Double
equation x y = x - y

eulerMethod :: (Double -> Double -> Double)
                -> (Double, Double)
                -> (Double, Double)
                -> Double
                -> [(Double, Double)]
eulerMethod equation initialCondition interval stepSize = iterate step initialCondition
  where
    step (x, y) = (x + stepSize, y + stepSize * equation x y)
```

Example for Test:

```
main :: IO ()
main = do
    let initialCondition = (0, 1)
        interval = (0, 2)
        stepSize = 0.1

        solution = takeWhile (\(x, _) -> x <= snd interval)
                    $ eulerMethod equation initialCondition interval stepSize
        result = head $ dropWhile (\(x, _) -> x < 0.5) solution

    putStrLn $ "Result at x = " ++ show (fst result) ++ ", y = " ++ show (snd result)
```

Output:

```
Result at x = 0.5: x = 0.5, y = 0.68098
```

(3)

This question is basically the same as the previous one, just change the formula and calculate it a few more times.

```
equation :: Double -> Double -> Double
equation x y = x + y

rungeKuttaMethod :: (Double -> Double -> Double)
                 -> (Double, Double)
                 -> (Double, Double)
                 -> Double
                 -> [(Double, Double)]
rungeKuttaMethod equation initialCondition interval stepSize = iterate step initialCondition
  where
    step (x, y) = let
        k1 = stepSize * equation x y
        k2 = stepSize * equation (x + 0.5 * stepSize) (y + 0.5 * k1)
        k3 = stepSize * equation (x + 0.5 * stepSize) (y + 0.5 * k2)
        k4 = stepSize * equation (x + stepSize) (y + k3)
        in (x + stepSize, y + (k1 + 2*k2 + 2*k3 + k4) / 6)
```

Example for Test:

```
main :: IO ()
main = do
    let equation x y = x + y
        initialCondition = (0, 1)
        interval = (0, 2)
        stepSize = 0.1

        solution = takeWhile (\(x, _) -> x <= snd interval)
                    $ rungeKuttaMethod equation initialCondition interval stepSize
        result = head $ dropWhile (\(x, _) -> x < 0.5) solution

    putStrLn $ "Result at x = " ++ show (fst result) ++ ", y = " ++ show (snd result)
```

Output:

```
Result at x = 0.5: x = 0.5, y = 1.7974412771936765
```